Assignment 4

– Solutions –

Total: 25 points

[3 points]

(1) a) Show ~~$\$$~~ R is reflexive.

Let $f \in F$ arbitrary. Then, since $f = 1 \cdot f$ and $1 \in \mathbb{R}^+$ it follows that $(f, f) \in F$

(1p) →

b) Show R is symmetric

1p → Let $f, g \in F$ arbitrary and assume $(f, g) \in R$. Then, there exist $c \in \mathbb{R}^+$ such that $f = cg$. Hence, $g = \frac{1}{c} f$ and $\frac{1}{c} \in \mathbb{R}^+$

Hence, $(g, f) \in R$.

c) R is transitive

1p → Let $f, g, h \in F$ arbitrary. Assume that $(f, g) \in R$ and $(g, h) \in R$

Then, there exist $c, d \in \mathbb{R}^+$ such that $f = cg$ and $g = dh$.

Then, $f = cg = c(dh) = (cd)h$ and $cd \in \mathbb{R}^+$.

Hence, $(f, h) \in R$.

(2) a) Equivalence classes of R on A.

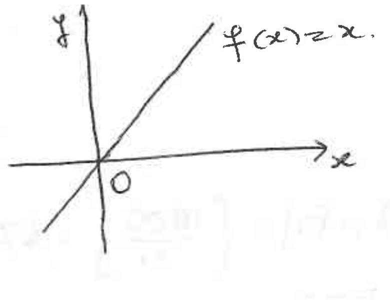
1p → $\{(-4,-3), (-1,0)\}$ $\qquad$ $\{(-2,-2)\}$

$\{(-2,-3), (0,-1), (1,0)\}$ $\qquad$ $\{(2,-3), (3,-2), (4,-1)\}$

b) $[(1,1)]_R = \{(x,y) \in \mathbb{R}^2 : x - y = 1 - 1 = 0\}$

1p → $= \{(x,y) \in \mathbb{R}^2, y = x\}$.

~~$\$$~~ The set of points ~~as th~~ can be represented as the points on the line $y = x$



③ [5 points]

a) The total no of arrangements of 5 out of 12 people in a row

1p → $P(12,5) = 12\cdot 11\cdot 10\cdot 9\cdot 8 = 95040$

The no of arrangements of 5 out of 12 people such that neither the bride nor the groom are among the 5

1p → $P(10,5) = 10\cdot 9\cdot 8\cdot 7\cdot 6 = 30240$

1p → The no of arrangements of 5 out of 12 people such that at least one of the bride and groom is among them

$P(12,5) - P(10,5) = \frac{12!}{7!} - \frac{10!}{5!} = 64800$

b) The procedure can be broken down into the following tasks:

1p →
- T_1: arrange 3 out of 10 people in a row ⤳ $P(10,3)$ ways
- T_2: arrange the bride and groom in a row ⤳ 2 ways
- T_3: choose 2 out of 5 spots for the bride and groom and merge the two rows accordingly ⇒ $\binom{5}{2}$ ways

1p → Total, by product rule: $P(10,3)\cdot 2\cdot\binom{5}{2} = \frac{10!}{7!}\cdot 2\cdot \frac{5!}{2!3!} = 14440$

[3 points]

④ Let $\mathcal{U} = \{1, 2, \dots, 1000\}$. Define $A = \{n\in\mathcal{U} : 3|n\}$

$B = \{n\in\mathcal{U} : 7|n\}$

1p → Need to find: $|\overline{A\cup B}|$

We have (using the Principle of Inclusion Exclusion)

$|\overline{A\cup B}| = |\mathcal{U}| - |A\cup B| = |\mathcal{U}| - |A| - |B| + |A\cap B|$

In addition, $|\mathcal{U}| = 1000$

$|A| = \left[\frac{1000}{3}\right] = 333$

1p → $|B| = \left[\frac{1000}{7}\right] = 142$

Also, $A\cap B = \{n\in\mathcal{U} : 21|n\}$ and so $|A\cap B| = \left[\frac{1000}{21}\right] = 47$

1p → Therefore, $|\overline{A\cup B}| = 1000 - 333 - 142 + 47 = 572$

[3 points]

(5) Note that $M_{ij} = \left(\frac{x_i + x_j}{2}, \frac{y_i + y_j}{2} \right)$

Therefore, the point M_{ij} has integer coordinates if and only if: $\frac{x_i + x_j}{2} \in \mathbb{Z}$ and $\frac{y_i + y_j}{2} \in \mathbb{Z}$.

1p → Equivalently, M_{ij} has integer coordinates if and only if (x_i and x_j have same parity) and (y_i and y_j have same parity)

Any point in the plane ~~has to~~ can be placed in one of the following boxes

1p → x-coordinate even, y-coordinate even | x-coord even, y-coord odd | ~~x-coord~~ odd, y-coord even | x-coord odd, y-coord odd

~~Since~~ We arrange the 5 points in their corresponding box, according to the parity of its coordinate.

1p → By the Pigeon hole Principle, there must be a box that contains 2 points, i.e. there exist 2 points whose coordinates have same parity.

This implies that there exist a mid point (the one ~~so~~ obtained from the line segment connecting the two points obtained above) whose coordinates are integer numbe[rs]

[3 points]

(6) $$(2x-3y)^{250} = \sum_{i=0}^{250} \binom{250}{i} \cdot (2x)^{250-i} (-3y)^i$$

1p → $$= \sum_{i=0}^{250} \binom{250}{i} 2^{250-i} (-3)^i x^{250-i} y^i$$

1p → (a) For $i = 51$, the coefficient is
$$\binom{250}{51} 2^{199} (-3)^{51} = -\binom{250}{51} 2^{199} 3^{51}$$

1p → (b) Since $19 + 200 \neq 250$, the coefficient is 0.

[2 points]

(7) Let s - bit string of length 12 with at most ten 1s and at most four 0s.

1p → It follows that ~~k~~ s has at least two 0s and at most four 0s, i.e the number of 0s is in: {2,3,4}

The number of strings with such property is:

1p → $\binom{12}{2}+\binom{12}{3}+\binom{12}{4} = \frac{12\cdot 11}{2} + \frac{12\cdot 11\cdot 10}{3\cdot 2} + \frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2}$

$= 66 + 220 + 495$

$= 781$

[4 points]

(8) (a) Consider the following tasks:

T_1: choose a winner of Grand Prize ⇝ 100 ways

T_2: choose a winner of Second Prize ⇝ 99 ways

1p → T_3: choose a winner of three identical prizes ⇝ $\binom{98}{3}$ ways

1p → Total, by product rule: $100 \cdot 99 \cdot \binom{98}{3} = 9900 \cdot 152096$

$= 1505750400$

(b) T_1: choose a winner of Grand Prize from UO students and Second Prize from Carleton students ~~or~~ or vice-versa: ⇝ $50^2 + 50^2$ ways

T_2: choose the winner of third prize ⇝ 98 ways

1p → T_3: choose the winner of forth prize ⇝ 97 ways

T_4: choose the winner of fifth prize ⇝ 96 ways

1p → Total: $2 \cdot 50^2 \cdot 98 \cdot 97 \cdot 96 = 4562880000$